PDE :-

$$F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0$$

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}$$

$$F(x, y, z, p, q) = 0$$

$$a(x, y, z)\frac{\partial z}{\partial x} + b(x, y, z)\frac{\partial z}{\partial y} = c(x, y, z)$$

Quasi-linear. where a, b, c are the functions of x, y, z

e.g. $z\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = xz^2$

$$a(x, y)\frac{\partial z}{\partial x} + b(x, y)\frac{\partial z}{\partial y} = c(x, y, z)$$

Semilinear 1st order PDE

e.g. $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = x^2 + z^2$

$$a(x, y)\frac{\partial z}{\partial x} + b(x, y)\frac{\partial z}{\partial y} = c(x, y)\,z + d(x, y)$$

1st order linear PDE.

e.g $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} - z = e^x$

$x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} - z = 0$ (1st order linear homogeneous PDE)

- **Geometrical interpretation of 1st order PDE :-**

Let $a(x, y, z)\dfrac{\partial z}{\partial x} + b(x, y, z)\dfrac{\partial z}{\partial y} - c(x, y, z) = 0$ --- (i)

be the Quasi-linear equation

We assume that the possible solution of (i) in the form $z = z(x, y)$ or is an implicite form $f(x, y, z) = z(x, y) - z = 0$ -- (ii) represents a possible solution surface or integral surface in (x, y, z) space. This is often called the integral surface of equation (i)

At any point (x, y, z) on the surface the gradient vector ∇f =

$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (z_x, z_y, -1)$$

this is the normal to the solution surface.

equation (i) can be re-written as.

$$a z_x + b z_y - c = (a, b, c).(z_x, z_y, -1) = 0$$

This clearly shows that the vector (a, b, c) must be tangent vector of the integral surface (ii) at any point (x, y, z) and here it is determined a direction field called characteristic direction for Monge axis. The's direction of fundamental important in determining of the solution of equation (i)

A curve in (x, y, z) space whose tangent at every point co-incide with the characteristic direction field (a, b, c) is called a a characteristic curve.

If the Parametric equation of the characteristic curve is given by $x = x(t)$, $y = y(t)$, $z = z(t)$

Then the tangent vector to the curve is $= \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$ which must be equal to (a, b, c).

$$\frac{dx}{dt} = a(x, y, z)$$

$$\frac{dy}{dt} = b(x, y, z)$$

$$\frac{dz}{dt} = c(x, y, z)$$

This are called the characteristic equation of the Quasi-linear equation.

- The general solution of linear PDE $Pp + Qq = R$ can be written in the form $F(u, v) = 0$, where F is an arbitary function and u, v are solution of the equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Find the general solution of linear PDE.

$$x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = (x+y) z$$

Lagrange's Auxiliary equations are..

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y) z}$$

Taking 1st and 2nd ratio we get

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

integrating we get.

$$-\frac{1}{x} = -\frac{1}{y} + c_1$$

$$\Rightarrow \frac{1}{x} - \frac{1}{y} = -c_1$$

$$\Rightarrow \frac{x-y}{xy} = c_1 = u$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dx - dy}{x^2 - y^2} = \frac{dz}{(x+y)z}$$

$$\therefore \frac{d(x-y)}{x^2-y^2} = \frac{dz}{(x+y)z}$$

$$\int \frac{d(x-y)}{x-y} = \int \frac{dz}{z}$$

$$\log(x-y) = \log z + \log c_2$$

$$\Rightarrow \frac{x-y}{z} = c_2 = v$$

$\therefore$ The required general solution is..

$$f\left(\frac{x-y}{xy}, \frac{x-y}{z}\right) = 0$$

where f is an arbitary function.

• $(y-z)\frac{\partial u}{\partial x} + (z-x)\frac{\partial u}{\partial y} + (x-y)\frac{\partial u}{\partial z} = 0$

$\Rightarrow$ Lagrange's Auxiliary equations are..

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} = \frac{du}{0}$$

Taking multipliers 1, 1, 1, each ratio

$$= \frac{dx + dy + dz}{y-z+z-x+x-y} = \frac{d(x+y+z)}{0}$$

$\therefore\ d(x+y+z) = 0 \Rightarrow x+y+z = c_1$

Again taking multipliers x, y, z, each ratio..

$$= \frac{x\,dx + y\,dy + z\,dz}{xy - xz + yz - xy + xz - yz} = \frac{x\,dx + y\,dx + z\,dx}{0}$$

$\therefore\ x\,dx + y\,dy + z\,dx = 0$

integrating ----- $x^2 + y^2 + z^2 = c_2$

Again. taking last ratio we get..

$du = 0$

$\Rightarrow u = c_3$

Hence the required general solution is..

$$u = f(x+y+z,\ x^2+y^2+z^2)$$

$y^2 p + xyq = x(z-2y)$

$\Rightarrow$ Lagrange's Auxiliary equations are..

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

taking 1st and 2nd ratio. we get..

$$\frac{dx}{y^2} = -\frac{dy}{xy}$$

$$\Rightarrow \int x\,dx + \int y\,dy = 0$$

$$\Rightarrow \frac{1}{2}(x^2+y^2) = \frac{c_1}{2}$$

$$\Rightarrow x^2+y^2 = c_1 = u$$

$$\frac{dx}{y} = \frac{dy}{-x}.$$

$$-x\,dx = y\,dy.$$

taking 2nd and 3rd ratio we get

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$\Rightarrow -\frac{dy}{y} = \frac{dz}{(z-2y)}$$

$$\Rightarrow \frac{dz}{dy} = -\frac{z-2y}{y}$$

$$\Rightarrow \frac{dz}{dy} + \frac{1}{y} z = 2$$

which is linear in z and y.

$$\therefore \text{ I.F.} = e^{\int 1/y\,dy} = e^{\log y} = y$$

$\therefore$ The solution is ----

$$z \cdot y = \int 2y\,dy \mp c_2$$

$$\Rightarrow zy = y^2 - c_2$$

$$\Rightarrow y^2 - yz = c_2 = v$$

$\therefore$ The required general solution is ...

$$f(x^2+y^2,\ y^2-yz)=0$$

- $(y+zx)p-(x+yz)q=x^2-y^2$

$\Rightarrow$ Lagrange's Auxiliary equation is ...

$$\frac{dx}{y+zx}=\frac{dy}{-(x+yz)}=\frac{dz}{x^2-y^2}$$

choosing $x, y, -z$ as multipliers, each fraction ..

$$\frac{x\,dx+y\,dy-z\,dz}{x(y+zx)-y(x+yz)-z(x^2-y^2)}=\frac{x\,dx+y\,dy-z\,dz}{0}$$

$\therefore\ x\,dx+y\,dy-z\,dz=0$

$\Rightarrow\ x^2+y^2-z^2=c_1=u$

choosing $y, x, 1$ as multipliers, each fraction

$$\frac{y\,dx+x\,dy+dz}{y(y+zx)-x(x+yz)+x^2-y^2}=\frac{y\,dx+x\,dy+dz}{0}$$

$\therefore\ y\,dx+x\,dy+dz=0$

$\int d(xy+z)=0$

$\Rightarrow\ xy+z=c_2=v$

$\therefore$ The required general solution is ..

$$f(x^2+y^2-z^2,\ xy+z)=0$$

- Find the integral surface of the linear PDE

$$x(y^2+z)p-y(x^2+z)q=(x^2-y^2)z$$

containing the straight line $x+y=0,\ z=1$

Lagrange's A E is..

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z} \quad --(i)$$

each ratio of (i) .

$$\frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{y^2+z-(x^2+z)+x^2-y^2} = \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}$$

$$\therefore \frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z} = 0$$

$$\Rightarrow \log x + \log y + \log z = \log c_1$$

$$\Rightarrow \log(xyz) = \log c_1$$

$$\Rightarrow xyz = c_1 \quad ...(ii)$$

taking multipliers $x, y, -1$, each ratio

$$\frac{x\,dx+y\,dy-dz}{x^2(y^2+z)-y^2(x^2+z)-(x^2-y^2)z} = \frac{x\,dx+y\,dy-dz}{0}$$

$$\therefore x\,dx+y\,dy-dz = 0$$

integrating.. $x^2+y^2-2z = c_2 \quad ...(iii)$

Now, the straight line in the parametric form:

$$x = t,\quad y = -t,\quad z = 1$$

From (ii), we get, $-t^2 = c_1$

From (iii) we get.. $2t^2-2 = c_2$

$\therefore$ The required integral surface can be obtained as.. $-2c_1-2 = c_2$

$$\Rightarrow -2(xyz)-2 = x^2+y^2-2z$$

$$\Rightarrow 2xyz+x^2+y^2-2z+2 = 0$$

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• Find the integral surface of the linear PDE

$$xp - yq = z$$

which contains the circle $x^2+y^2 = 1,\ z = 1$

⇒ Lagrange's A.E. is .

$$\frac{dx}{x} = -\frac{dy}{y} = \frac{dz}{z} \quad \text{--- (i)}$$

Taking 1st and 2nd ratio we get -

$$\frac{dx}{x} = -\frac{dy}{y}$$

$$\Rightarrow \log x + \log y = \log c_1$$

$$\Rightarrow xy = c_1 \quad \text{--- (ii)}$$

Taking 2nd and 3rd ratio we get . .

$$-\frac{dy}{y} = \frac{dz}{z}$$

$$\Rightarrow \log y = -\log z + \log c_2$$

$$\Rightarrow yz = c_2 \quad \text{--- (iii)}$$

Now, the parametric equation of the circle be . .

$$x = t,\quad y = \sqrt{1-t^2},\quad z = 1$$

From (ii) we get . . $t\sqrt{1-t^2} = c_1$

From (iii) we get . . $\sqrt{1-t^2} = c_2$

$$\therefore t = \cancel{c_1 c_2}\ \ c_1/c_2$$

$$\therefore \cancel{c_2 = \sqrt{1-(c_1c_2)^2}}$$

$$\Rightarrow \cancel{c_2^2 = 1 - c_1^2 c_2^2}$$

$$\Rightarrow \cancel{y^2z^2 = 1 - (xy^2z)^2}$$

$$\Rightarrow \cancel{x^2y^4z^2 + y^2z^2 = 1}$$

$$\Rightarrow \cancel{y^2z^2(1+x^2y^2) = 1}$$

$$c_2 = \sqrt{1 - \frac{c_1^2}{c_2^2}}$$

$$c_2^2 = \sqrt{c_2^2 - c_1^2}$$

$$c_2^4 = c_2^2 - c_1^2$$

$$c_1^2 = c_2^2(1 - c_2^2)$$

$$x^2y^2 = y^2z^2(1 - y^2z^2)$$

$xp + yq = z$

$x/y = c_1$

$y/z = c_2$

$\frac{x}{\sqrt{1-x^2}} = c_1$

$\sqrt{1-x^2} = c_2$

$x = c_1 c_2$

Find the integral surface of the linear PDE.

$$(x-y)p + (y-x-z)q = z$$

which contains the circle $x^2+y^2=1,\ z=1$

$\Rightarrow$ Lagrange's A.E. is..

$$\frac{dx}{x-y} = \frac{dy}{y-x-z} = \frac{dz}{z} \quad \text{--- (i)}$$

choosing 1, 1, 1 as multipliers, each ratio

$$\frac{dx+dy+dz}{x-y+y-x-z+z} = \frac{d(x+y+z)}{0}$$

$$d(x+y+z) = 0$$

$$\Rightarrow \quad x+y+z = c_1 \quad \text{--- (ii)}$$

Taking 2nd and 3rd ratio we get

$$\frac{dy}{y-x-z} = \frac{dz}{z}$$

$$\frac{dy}{y-(c_1-y)} = \frac{dz}{z}$$

$$\Rightarrow \quad \cancel{2y}\int \frac{2\,dy}{2y-c_1} - \int \frac{2\,dz}{z} = 0$$

$$\Rightarrow \quad \log(2y-c_1) - \log z^2 = \log c_2$$

$$\Rightarrow \quad \frac{2y-x-y-z}{z^2} = c_2 \quad \Rightarrow \quad \cancel{2}(y-x-z)/z^2 = c_2 \quad \text{--- (iii)}$$

$$\frac{d(x-y-z)}{2(x-y-z)} = \frac{dz}{dx}$$

$$\frac{1}{2}\log(x-y-z) = \log z^2$$

The circle is given by .. $x^2+y^2=1,\quad z=1$

putting $z=1$ in (ii) and (iii) we get..

$$x+y = c_1-1 \quad \text{and} \quad y-x = c_2+1$$

But $\quad 2(x^2+y^2) = \cancel{2}\,(x+y)^2 + (y-x)^2$

$$\Rightarrow \quad 2 = (c_1-1)^2 + (c_2+1)^2$$

$$\Rightarrow \quad c_1^2 + c_2^2 - 2c_1 + 2c_2 = 0$$

$$\Rightarrow \quad (x+y+z)^2 + \frac{(y-x-z)^2}{z^4} - 2.(x+y+z) + 2\frac{(y-x-z)}{z^2} = 0$$

$$\Rightarrow \quad z^4(x+y+z)^2 + (y-x-z)^2 - 2z^4(x+y+z) + 2z^2(y-x-z) = 0$$

which is the required integral surface

- **Compatible system of 1st order PDE :-**

Def :- Two first order PDE are said to be compatible if they have common solution.

- **Necessary and sufficient condition for compatibility of 1st order PDE :-**

let $f(x, y, z, p, q) = 0$ — (i)

$g(x, y, z, p, q) = 0$ -- (ii)

be compatable and also ∃ det $J = \frac{\partial(f,g)}{\partial(p,q)} \neq 0$

since (i) and (ii) have common solution, we can solve them and obtain explicite expression for p and q in the form

$$\left.\begin{aligned} p &= \phi(x, y, z) \\ q &= \psi(x, y, z) \end{aligned}\right\} \text{---- (iii)}$$

and $dz = p\,dx + q\,dy$

$$\Rightarrow \quad \phi(x, y, z)\,dx + \psi(x, y, z)\,dy = dz \quad \text{--- (iv)}$$

should be integrable for which the necessary condition is.

$$\vec{X} \cdot \text{curl}\,\vec{X} = \vec{0} \qquad \text{where } X = \{\phi, \psi, -1\}$$

i.e $\{\phi, \psi, -1\} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi & \psi & -1 \end{vmatrix} = 0$

$\Rightarrow \phi(-\psi_z) + \psi(\phi_z) = \psi_x - \phi_y$

$\Rightarrow \psi_x + \phi\psi_z = \phi_y + \psi\phi_z$ --- (v)

differentiating (i) wrt x and z we have:

$$f_x + f_p \frac{\partial p}{\partial x} + f_q \cdot \frac{\partial q}{\partial x} = 0$$

$$f_z + f_p \frac{\partial p}{\partial z} + f_q \frac{\partial q}{\partial z} = 0$$

$$\Rightarrow f_x + f_p \frac{\partial \phi}{\partial x} + f_q \frac{\partial \psi}{\partial x} = 0$$

$$f_z + f_p \frac{\partial \phi}{\partial z} + f_q \frac{\partial \psi}{\partial z} = 0$$

$$\Rightarrow f_x + f_p \phi_x + f_q \psi_x = 0 \quad \text{--- (vi)}$$

$$f_z + f_p \phi_z + f_q \psi_z = 0 \quad \text{--- (vii)}$$

From (vi) and (vii) we can write

$$(f_x + \phi f_z) + f_p(\phi_x + \phi\phi_z) + f_q(\psi_x + \phi\psi_z) = 0 \quad \text{--- (viii)}$$

similarly from (ii) we get

$$(g_x + \phi g_z) + g_p(\phi_x + \phi\phi_z) + g_q(\psi_x + \phi\psi_z) = 0 \quad \text{--- (ix)}$$

$$\frac{\psi_x + \phi\psi_z}{f_p(g_x + \phi g_z) - g_p(f_x + \phi f_z)} = \frac{1}{-f_p g_q + f_q g_p} = -\frac{1}{J}$$

$$\therefore \psi_x + \phi\psi_z = -\frac{1}{J}\left[\frac{\partial(f,g)}{\partial(x,p)} + \phi\frac{\partial(f,g)}{\partial(z,p)}\right] \quad \text{-- (x)}$$

~~similarly~~ differentiating (i) wrt y and z we have.

$$\phi_y + \psi\phi_z = \frac{1}{J}\left[\frac{\partial(f,g)}{\partial(y,q)} + \psi\frac{\partial(f,g)}{\partial(z,q)}\right] \quad \text{-- (xi)}$$

~~using~~ (v), equating (x) and (xi) we get

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$$\frac{\partial(f,g)}{\partial(x,p)} + \phi \frac{\partial(f,g)}{\partial(z,p)} = -\frac{\partial(f,g)}{\partial(y,q)} - \psi \frac{\partial(f,g)}{\partial(z,q)}$$

$$\Rightarrow \frac{\partial(f,g)}{\partial(x,p)} + p \frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q \frac{\partial(f,g)}{\partial(z,q)} = 0$$

This is the desired condition for compatibility.

- Show that the PDE's $xp = yq$

$$z(xp+yq) = 2xy$$

are compatable and hence find its solution.

$\Rightarrow$ $f(x,y,z,p,q) = xp - yq = 0$ —— (i)

$g(x,y,z,p,q) = z(xp+yq) - 2xy = 0$ -- (ii)

$$\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial p} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} p & pz-2y \\ x & zx \end{vmatrix} = pzx - xpz + 2xy = 2xy$$

$$\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial g}{\partial z} \\ \frac{\partial f}{\partial p} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} 0 & px+qy \\ x & xz \end{vmatrix} = -px^2 - qxy$$

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial g}{\partial y} \\ \frac{\partial f}{\partial q} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} -q & zq-2x \\ -y & yz \end{vmatrix} = -qyz + yzq - 2xy = -2xy$$

$$\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial g}{\partial z} \\ \frac{\partial f}{\partial q} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & xp+yq \\ -y & yz \end{vmatrix} = pxy + qy^2$$

Now, $\frac{\partial(f,g)}{\partial(x,p)} + p \frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q \frac{\partial(f,g)}{\partial(z,q)}$

$= 2xy + p(-px^2 - qxy) - 2xy + q(pxy + qy^2)$

$= -p^2x^2 + q^2y^2 = 0 \quad [\because xp = yq]$

$\therefore$ The PDE's are compatable.

Solving (i) and (ii) we get.

$$q = \frac{x}{z}, \quad p = \frac{y}{z}$$

putting the values of p and q in the explicite equation $dz = p\,dx + q\,dy$ we get.

$$dz = \frac{y}{z}dx + \frac{x}{z}dy$$

$$\int z\,dz = \int d(xy)$$

$$z^2 = 2xy + c$$

- Show that the PDE's $xp - yq = x$

$$x^2p + q = xz$$

are compatable and find its solution.

$\Rightarrow$ $f(x, y, z, p, q) = xp - yq - x = 0$ --- (i)

$g(x, y, z, p, q) = x^2p + q - xz = 0$ --- (ii)

$x^2p - xz + \frac{xp - x}{y} = 0.$

$\Rightarrow p\left(x^2 + \frac{x}{y}\right) = xz + \frac{x}{y}$

$\Rightarrow p = \frac{xyz + x}{x^2y + x}.$

$= \frac{yz + 1}{1 + xy}$

$$\frac{\partial(f, g)}{\partial(x, p)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial p} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} p-1 & 2xp - z \\ x & x^2 \end{vmatrix} = px^2 - x^2 - 2px^2 + xz = -px^2 - x^2 + xz$$

$$\frac{\partial(f, g)}{\partial(z, p)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial g}{\partial z} \\ \frac{\partial f}{\partial p} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} 0 & -x \\ x & x^2 \end{vmatrix} = x^2$$

$$\frac{\partial(f, g)}{\partial(y, q)} = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial g}{\partial y} \\ \frac{\partial f}{\partial q} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} -q & 0 \\ -y & 1 \end{vmatrix} = -q$$

$$\frac{\partial(f, g)}{\partial(z, q)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial g}{\partial z} \\ \frac{\partial f}{\partial q} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & -x \\ -y & 1 \end{vmatrix} = -xy$$

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Now, $\frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)}$

$= -p\cancel{x^2} - x^2 + xz + p\cancel{x^2} - q - qxy$

$= -x^2 + xz - q - qxy$

$= -x^2 + x^2 p - qxy \quad [\because x^2 p + q = xz]$

$= x(px - x - qy)$

$= x(x - x) = 0 \quad [\because px - qy = x]$

$\therefore$ The PDE's are compatable.

Solving (i) and (ii) we get--

$$p = \frac{1+yz}{1+xy}, \qquad q = \frac{x(z-x)}{1+xy}$$

putting the values of p and q in the expliците equation $dz = p\,dx + q\,dy$ we get----

$$dz = \frac{1+yz}{1+xy}dx + \frac{x(z-x)}{1+xy}dy$$

$\Rightarrow$ ~~$(1+xy)dz = dx + yz\,dx + xz\,dy - x^2dy$~~

$\Rightarrow (1+xy)dz = (1+yz)dx + x(z-x)dy$

$\Rightarrow (1+xy)dz - z\,d(xy) = dx - x^2 dy$

$\Rightarrow \frac{(1+xy)dz - z\,d(xy)}{(1+xy)^2} = \frac{dx - x^2dy}{(1+xy)^2} = \frac{(dx/x^2) - dy}{(y+1/x)^2}$

$\Rightarrow \int d\left(\frac{z}{1+xy}\right) = -\int \frac{d(y+1/x)}{(y+1/x)^2}$

$\Rightarrow \frac{z}{1+xy} = \frac{1}{(y+1/x)} + c$

$\Rightarrow z - x = c(1+xy)$

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3. Show that the PDE's $p^2+q^2=1$

$$(p^2+q^2)x = pz$$

are compatable and find its solution

$\Rightarrow$ $f(x,y,z,p,q) = p^2+q^2-1=0 \quad \ldots (i)$

$g(x,y,z,p,q) = (p^2+q^2)x - pz = 0 \quad \text{---} (ii)$

$$\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial p} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} 0 & p^2+q^2 \\ 2p & 2px-z \end{vmatrix} = -2p(p^2+q^2)$$

$$\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial g}{\partial z} \\ \frac{\partial f}{\partial p} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} 0 & -p \\ 2p & 2px-z \end{vmatrix} = 2p^2$$

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial g}{\partial y} \\ \frac{\partial f}{\partial q} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 2q & 2qx \end{vmatrix} = 0$$

$$\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial g}{\partial z} \\ \frac{\partial f}{\partial q} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & -p \\ 2q & 2qx \end{vmatrix} = 2pq$$

Now, $\frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)}$

$= -2p(p^2+q^2) + p\cdot 2p^2 + 0 + q\cdot 2pq$

$= -2p^3 - 2pq^2 + 2p^3 + 2pq^2$

$= 0$

$\therefore$ The PDE's are compatable

solving (i) and (ii) we get

$$p = \frac{x}{z}, \quad q = \sqrt{1-\frac{x^2}{z^2}} = \frac{\sqrt{z^2-x^2}}{z}$$

putting the values of p and q in ~~eq~~ $dz = p\,dx + q\,dy$ we get

$$dz = \frac{x}{z}\,dx + \frac{\sqrt{z^2-x^2}}{z}\,dy$$

$$z\,dz - x\,dx = \sqrt{z^2-x^2}\,dy$$

$$\frac{1}{2}\,d(z^2-x^2) = \sqrt{z^2-x^2}\,dy$$

$$\int \frac{d(z^2-x^2)}{\sqrt{z^2-x^2}} = \int 2\,dy$$

$$2\sqrt{z^2-x^2} = 2y + 2c$$

$$z^2 - x^2 = (y+c)^2$$

$$\Rightarrow \quad z^2 = x^2 + (y+c)^2$$

- **Surfaces orthogonal to a given system of surfaces :-**

Suppose we are given one parameter family of surfaces characterised by the equation $f(x,y,z) = c$ --- (i) and we used to find system of surfaces which cuts each of this given surfaces at right - angled or orthogonally. The normal at the point (x,y,z) to the surface of the system (i) which pass through that points is the direction given by the drs $(P,Q,R) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ --- (ii).

If the surface with equation $z = \phi(x,y)$ --- (iii) cuts each surface of the given system ~~orth~~ orthogonally, then its normal at point (x,y,z) which is in the direction $\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)$ is the perpendicular to the direction (P,Q,R) of the normal to the surface (i) at the point (x,y,z).

i.e $\quad P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} = R$

or equivalently, $\quad \frac{\partial f}{\partial x}\cdot\frac{\partial z}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{\partial z}{\partial y} = R.$

Find the surface which intersect the surfaces of the system $z(x+y) = c(3z+1)$ orthogonally and which passes through the circle $x^2+y^2=1,\ z=1$

$\Rightarrow$ $f(x,y,z) = \dfrac{z(x+y)}{3z+1} = c$

$\therefore \dfrac{\partial f}{\partial x} = \dfrac{z}{3z+1}, \quad \dfrac{\partial f}{\partial y} = \dfrac{z}{3z+1}, \quad \dfrac{\partial f}{\partial z} = (x+y)\left[\dfrac{1}{(3z+1)^2}\right]$

The required orthogonal surface is solution of.

$$p\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}$$

$$\Rightarrow \frac{z}{3z+1}\cdot p + \frac{z}{3z+1}q = \frac{x+y}{(3z+1)^2}$$

$$\Rightarrow z(3z+1)p + z(3z+1)q = x+y \quad \text{--- (i)}$$

Lagrange's A.E is..

$$\frac{dx}{z(3z+1)} = \frac{dy}{z(3z+1)} = \frac{dz}{x+y}$$

Taking 1st and 2nd ratio we get.

$$\frac{dx}{z(3z+1)} = \frac{dy}{z(3z+1)}$$

$$\Rightarrow d(x-y) = 0$$

$$\Rightarrow x-y = c_1$$

$\therefore$, choosing $x, y, -z(3z+1)$ as multipliers, each ratio

$$= \frac{xdx+ydy - z(3z+1)dz}{0}$$

$$\therefore \int xdx + \int y\,dy - \int 3z^2dz - \int z\,dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} - z^3 - \frac{z^2}{2} = \frac{c_2}{2}$$

$$\Rightarrow x^2+y^2-2z^3-z^2 = c_2$$

Hence any surface which is orthogonal to (i) has equation of the form :—

$$x^2+y^2-2z^3-z^2 = \phi(x-y)$$

since the surface passing through the circle $x^2+y^2=1,\ z=1,$

$$\therefore \phi(x-y) = 1-2-1 = -2$$

Hence the required surface is.

$$x^2+y^2-2z^3-z^2 = -2$$

- **Charpit's Method :-**

Charpit's method is used to find the complete solution of linear PDE of the form $f(x,y,z,p,q)=0$ ... (i) The basic idea of Charpit's Method is the introduction of another PDE of 1st order of the form $g(x,y,z,p,q)=0$ ... (ii)

Then solving equation (i) and (ii) and substituting in $dz = p\,dx + q\,dy$ if the complete integral exists, then equation (iii) is the complete solution.

The main task of Charpit's Method is to find equation (ii) which is compatable with equation (i)

equation (ii) is compatable with equation (i), if

$$\frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = 0$$

$$\Rightarrow \left(\frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\cdot\frac{\partial g}{\partial x}\right) + p\left(\frac{\partial f}{\partial z}\,\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\cdot\frac{\partial g}{\partial z}\right)$$

$$+\left(\frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\,\frac{\partial g}{\partial y}\right) + q\left(\frac{\partial f}{\partial z}\cdot\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\cdot\frac{\partial g}{\partial z}\right) = 0$$

$$\Rightarrow -f_p\frac{\partial g}{\partial x} - f_q\frac{\partial g}{\partial y} - (pf_p + qf_q)\frac{\partial g}{\partial z} + (f_x + pf_z)\frac{\partial g}{\partial p} + (f_y + qf_z)\frac{\partial g}{\partial q} = $$

$$\Rightarrow f_p \frac{\partial g}{\partial x} + f_q \frac{\partial g}{\partial y} + (pf_p + qf_q)\frac{\partial g}{\partial z} - (f_x + pf_z)\frac{\partial g}{\partial p} - (f_y + qf_z)\frac{\partial g}{\partial q} = 0$$

This is a linear equation of first order, which the ~~ax~~ auxiliary function g of the equation (ii) must satisfy. Its integrals are integrals of the auxiliary equations.

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{(pf_p + qf_q)} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}.$$

The equation are known as Charpit's auxiliary equations.

Find the complete integral of -- $x^2p^2 + y^2q^2 - 4 = 0$

$$\Rightarrow f(x, y, z, p, q) = x^2p^2 + y^2q^2 - 4 = 0 \quad \text{--- (i)}$$

Charpit's A.E. are

$$\frac{dx}{2x^2p} = \frac{dy}{2y^2q} = \frac{dz}{p.2px^2 + q.2qy^2} = \frac{dp}{-(2xp^2+0)} = \frac{dq}{-(2yq^2+0)}$$

Taking 1st and 4th ratio we get.

$$\frac{dx}{2x^2p} = \frac{dp}{-2xp^2}$$

$$\Rightarrow \int\frac{dx}{x} + \int\frac{dp}{p} = 0$$

$$\Rightarrow \log x + \log p = \log a$$

$$\Rightarrow px = a$$

$$p = \frac{a}{x}$$

putting the value of p in (i) we get..

$$a^2 + y^2q^2 - 4 = 0$$

$$q = \frac{\sqrt{4-a^2}}{y}$$

Hence from $dz = p\,dx + q\,dy$ we get

$$\int dz = \int \frac{a}{x}\,dx + \int \frac{\sqrt{4-a^2}}{y}\,dy$$

$$z = a\log x + \sqrt{4-a^2}\log y + b$$

- $p + q = pq$

$\Rightarrow f(x, y, z, p, q) = p + q - pq = 0$ --- (i).

Charpit's A.E are

$$\frac{dx}{1-q} = \frac{dy}{1-p} = \frac{dz}{p(1-q)+q(1-p)} = \frac{dp}{0} = \frac{\cancel{dq}\,dq}{0}$$

Taking the 4th fraction...

$dp = 0 \Rightarrow p = a$

putting in (i) we get..

$$a + q - aq = 0$$

$$q(1-a) = -a$$

$$\Rightarrow q = \frac{a}{a-1}$$

Hence from $dz = p\,dx + q\,dy$, we get..

$$\int dz = \int a\,dx + \int \frac{a}{a-1}\,dy$$

$$\Rightarrow z = ax + \frac{a}{a-1}y + b$$

- $z^2(p^2z^2 + q^2) = 1$

$\Rightarrow f(x, y, z, p, q) = z^2(p^2z^2+q^2) - 1 = 0$ --- (i)

Charpit's A.E. are

$$\frac{dx}{2pz^4} = \frac{dy}{2qz^2} = \frac{dz}{2p^2z^4 + 2q^2z^2} = \frac{dp}{\cancel{(0+p\,p^2+z^3)}\ -p(4p^2z^3+2zq^2)} = \frac{dq}{\cancel{(0+q\ldots)}\ -q(4p^2z^3+2zq^2)}$$

Taking 4th and 5th ratio we get.

$$\frac{dp}{p} = \frac{dq}{q}$$

$$\Rightarrow p = aq$$

putting the value of p in (i) we get.

$$z^2 q^2 (a^2 z^2 + 1) = 1$$

$$q = \frac{1}{z\sqrt{a^2z^2+1}}$$

$$p = \frac{a}{z\sqrt{a^2z^2+1}}$$

Hence from $dz = p\,dx + q\,dy$ we get..

$$dz = \frac{a}{z\sqrt{a^2z^2+1}}\,dx + \frac{dy}{z\sqrt{a^2z^2+1}}$$

$$\int a\,dx + dy = \int z\sqrt{a^2z^2+1}\,dz$$

$$\Rightarrow ax + y + b = \frac{1}{3a^2}(a^2z^2+1)^{3/2}$$

$$\Rightarrow (a^2z^2+1)^3 = 9a^4(ax+y+b)^2$$

$$q(z+px)^2 = 1$$

$$\Rightarrow f(x, y, z, p, q) = q(z+px)^2 - 1 = 0 \quad \text{---(i)}$$

Charpit's A.E are

$$\frac{dx}{2q(z+px)x} = \frac{dy}{(z+px)^2} = \frac{dz}{p\,2qx(z+px) + q(z+px)^2} = \frac{dp}{-(q\,2(z+px)\,p + pq\,2(z+px))} = \frac{dq}{-(2q^2(z+px))}$$

Taking 1st and last ~~last~~ fraction we get..

$$\frac{dx}{2q(z+px)x} = \frac{dq}{-2q^2(z+px)}$$

$$\Rightarrow \frac{dx}{x} + \frac{dq}{q} = 0 \quad \Rightarrow qx = b$$

putting the value of q in (i) we get

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$$\frac{b}{x}(z+px)^2 = 1$$

$$zx + px = \frac{b}{\sqrt{a}}\sqrt{\frac{x}{b}}$$

$$\Rightarrow \cancel{1+p = \frac{1}{\sqrt{ax}}} \qquad \Rightarrow p = \frac{\sqrt{x}-\sqrt{b}\,z}{\sqrt{b}\,x}$$

$$\Rightarrow \cancel{p = \frac{1}{\sqrt{ax}}} - 1$$

Hence from $dz = p\,dx + q\,dy$ we get.

$$\cancel{z = \frac{1}{\sqrt{a}}} \qquad dz = \frac{\sqrt{x}-\sqrt{b}\,z}{\sqrt{b}\,x}\,dx + \frac{b}{x}\,dy$$

$$x\,dz = \frac{1}{\sqrt{b}}\sqrt{x}\,dx - z\,dx + b\,dy$$

$$x\,dz + z\,dx = \frac{1}{\sqrt{b}}\sqrt{x}\,dx + b\,dy$$

$$\int d(xz) = \int \frac{1}{\sqrt{b}}\sqrt{x}\,dx + \int b\,dy$$

$$xz = \frac{2}{3\sqrt{b}}\,x\sqrt{x} + by + c$$

$$\Rightarrow xz = 2a\,x\sqrt{x} + by + c$$

- $pxy + pq + qy = yz$

$$\Rightarrow f(x,y,z,p,q) = pxy + pq + qy - yz$$

Charpit's A.E. are

$$\frac{dx}{xy+q} = \frac{dy}{p+y} = \frac{dz}{p(xy+q)+q(p+y)} = \frac{dp}{-(py+p(-y))} = \frac{dq}{-(px+q-z+q(-y))}$$

Taking 4th fraction, we get

$$dp = 0 \Rightarrow p = a$$

$$axy + aq + qy = yz$$

$$q(y+a) = yz - axy$$

$$q = \frac{y(z-ax)}{y+a}$$

putting the values of p and q in $dz = pdx + qdy$, we get..

$$dz = a\,dx + \frac{y(z-ax)}{y+a}\,dy$$

$$\frac{d(z-ax)}{z-ax} = \frac{y\,dy}{y+a}.$$

$$\Rightarrow \int \frac{d(z-ax)}{z-ax} = \int dy - \int \frac{a}{y+a}\,dy$$

$$\Rightarrow \log(z-ax) = y - a\log(y+a) + \log b$$

$$\Rightarrow \log(z-ax) + \log(y+a)^a = y + \log b$$

$$\Rightarrow (z-ax)(y+a)^a = be^y$$

- $p^2x + q^2y = z$

$$\Rightarrow f(x, y, z, p, q) = p^2x + q^2y - z = 0 \quad \text{-- (i)}$$

Charpit's A.E are

$$\frac{dx}{2px} = \frac{dy}{2qy} = \frac{dz}{p(2px) + q\cdot(2qy)} = \frac{dp}{-(p^2 + p(-1))} = \frac{dq}{-(q^2 + q(-1))} \quad \text{-- (ii)}$$

~~Taking last two ratio we get~~

Now, each fraction in (ii)..

$$\frac{2px\,dp + p^2dx}{2px\,(p - p^2) + p^2\cdot 2px} = \frac{2qy\cdot dq + q^2dy}{2qy\cdot \cancel{2qy}\,(q-q^2) + q^2\cdot 2qy}$$

$$\Rightarrow \frac{d(p^2x)}{2p^2x} = \frac{d(q^2y)}{2q^2y}$$

integrating ... $\log(p^2 x) = \log(q^2 y) + \log a$

$$\Rightarrow p^2 x = a y q^2$$

$\because$ From (i) we get.

$$a y q^2 + q^2 y = z$$

$$q = \frac{\sqrt{z}}{\sqrt{1+a}\,\sqrt{y}}$$

$$\therefore \quad p = \left(\frac{ay}{x}\right)^{1/2} \cdot q$$

$$\Rightarrow p = \frac{\sqrt{ay}}{\sqrt{x}} \cdot \frac{\sqrt{z}}{\sqrt{1+a}\,\sqrt{y}} = \frac{\sqrt{a}\sqrt{z}}{\sqrt{1+a}\,\sqrt{x}}$$

Now, from $dz = p\,dx + q\,dy$ we get.

$$\Rightarrow dz = \frac{\sqrt{a}\sqrt{z}}{\sqrt{1+a}\,\sqrt{x}}\,dx + \frac{\sqrt{z}}{\sqrt{1+a}\cdot\sqrt{y}}\,dy.$$

$$\Rightarrow \int (1+a)^{1/2} \frac{dz}{\sqrt{z}} = \int \sqrt{a}\,\frac{dx}{\sqrt{x}} + \int \frac{dy}{\sqrt{y}}$$

$$\Rightarrow (1+a)^{1/2} \cdot 2\sqrt{z} = 2\sqrt{a}\sqrt{x} + 2\sqrt{y} + 2b$$

$$\Rightarrow (1+a)^{1/2}\sqrt{z} = \sqrt{a}\cdot\sqrt{x} + \sqrt{y} + b$$

- $(p+q)(px+qy) = 1$

$$\Rightarrow f(x,y,z,p,q) = (p+q)(px+qy) - 1$$

Charpit's A.E. are ...

$$\frac{dx}{2px+qy+qx} = \frac{dy}{py+px+2qy} = \frac{dz}{p(2px+qy+qx) + q(py+px+2qy)} = \frac{dp}{-(p^2+pq)} = \frac{dq}{-(q^2+pq)}$$

taking last two fractions we get.

$$\frac{dp}{p^2+pq} = \frac{dq}{q^2+pq}$$

$$\Rightarrow \int \frac{dp}{p} = \int \frac{dq}{q}$$

$$\Rightarrow \log p = \log q + \log a$$

$$\Rightarrow p = aq$$

$$\therefore (aq+q)(aqx+qy) = 1$$

$$q^2 (1+a)(ax+y) = 1$$

$$\Rightarrow q = \frac{1}{(1+a)^{1/2} (ax+y)^{1/2}}$$

$$\therefore p = \frac{a}{(1+a)^{1/2} (ax+y)^{1/2}}$$

Now from $dz = p\,dx + q\,dy$ we get.

$$dz = \frac{a\,dx}{(1+a)^{1/2} (ax+y)^{1/2}} + \frac{dy}{(1+a)^{1/2} (ax+y)^{1/2}}$$

$$\Rightarrow (1+a)^{1/2} \int dz = \int \frac{d(ax+y)}{\sqrt{ax+y}}$$

$$\Rightarrow (1+a)^{1/2} z = 2(ax+y)^{1/2} + b$$

- Find the complete integral of $(p^2+q^2)x = pz$

$$\Rightarrow f(x, y, z, p, q) = (p^2+q^2)x - pz = 0 \quad \text{(i)}$$

Charpit's A.E are

$$\frac{dx}{2px - z} = \frac{dy}{2qx} = \frac{dz}{p(2px-z) + q \cdot 2qx} = \frac{dp}{-(p^2+q^2+p(-p))} = \frac{dq}{-(0+q(-p))}$$

Taking last two ratio we get..

$$\frac{dp}{q^2} = \frac{dq}{-pq}$$

$$\Rightarrow \frac{dp}{q} = -\frac{dq}{p}$$